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Equivalence of q-bosons using the exponential phase operator

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Abstract. Various forms of the q-boson are explained and their hidden symmetry revealed by transformations using the exponential phase operator. Both the one-component and the multicomponent q-bosons are discussed. As a byproduct, we obtain a new boson algebra having a shifted vacuum structure and define a global operator $U(1)$ gauge transformation.

1. Introduction

The q-boson‡ is the simplest q-deformed algebra having a deformation parameter. Various kinds of q-bosons have been introduced for their own motives. They can be transformed into each other by proper redefinitions of their generators. The existence of such transformations suggests that the q-bosons may be equivalent to each other, and there will exist underlying symmetries that make the equivalence sensible. However, we cannot see any clues for finding such symmetries.

One of the reasons for this fallacy of understanding may be connected to the difference in the methods of construction, such as in the process of the Schwinger realization of $su_q(2)$ in [1–3], and of $su_q(n)$ in [4], as the components of the fundamental representation of $su_q(n)$ [5], or for other motives [5–10]. Each type of q-bosons transforms into the others [11–13].

Another reason is that the Hopf algebraic structure does not fix the q-boson uniquely [14, 15]. However, in this paper we will not be concerned about the Hopf structure [16, 17].

Recently, a new method of constructing q-bosons has been presented in [18, 19] in which q-bosons are rewritten by the bi-product of different generators of bosonic type and taking an expectation value for a density operator. Although its connection with the Hopf structure has not been clarified, it explains some q-properties easily.

In this paper, we restrict the method to the boson and the exponential phase operator [20–22]. Depending on the explicit choice, the different forms of q-bosons appear, clearly pointing to the difference and similarity between the q-bosons. As a result, q-bosons, treated in this paper, are related to each other with the operator version of $U(1)$ gauge transformation and the similarity transformation by the exponential phase operator. As a byproduct, we find a new boson algebra which is equivalent to the normal boson algebra except of the shifted vacuum, and the operator version of the $U(1)$ gauge transformation

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‡ There are equivalent terminologies such as the q-Heisenberg-Weyl algebra and the q-oscillator, but we take the q-boson in order to emphasize its relation with the complex variable.

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which becomes a normal gauge transformation in the classical limit. We work both on the one-component q-bosons and on the multicomponent forms.

The paper is arranged as follows. In section 2, we treat the statistical mixture in order to see the physical meaning of the exponential phase operator and to define the physical density operator for q-bosons. In section 3, we construct q-bosons and find connections between them. We also define a new boson algebra and the operator version of the $U(1)$ gauge transformation. In section 4, we treat the multi-component q-bosons and find a hidden symmetry. Finally, we briefly discuss problems encountered along with extensions to the q-fermion and so on.

2. The statistical mixture and the Cuntz algebra

Let us consider the statistical mixture of the bosonic system in order to define the density operator of the expectation value which will appear in the process of constructing q-bosons. We also discuss the physical meaning of the exponential phase operator. A more detailed physical system related to this section should be found in quantum optics [20].

Assume that a system has a known probability P_R when it is in the state $|R\rangle$. Here, R is a label that runs over a set of pure states sufficient to describe the system. The states described by the probability P_R are called the *statistical mixture*, and the magnitude of the P_R for $|R\rangle$ contains all the available information about the state. An example of a statistical mixture is provided by the thermal excitation of the photons in a cavity mode. The probability P_n means that n photons are excited at the temperature T. In this case, the result of an experiment depends on an ensemble average of some observable quantity.

Consider some quantum mechanical operator O. The average value of the observable for the pure state $|R\rangle$ is $\langle R|O|R\rangle$, and hence the ensemble average of the observable for the statistical mixture specified by P_R is

$$\langle O\rangle = \sum_R P_R \langle R|O|R\rangle. \tag{2.1}$$

It will be assumed that P_R is a normalized probability distribution

$$\sum_R P_R = 1. \tag{2.2}$$

The average $\langle O\rangle$ is independent of the particular complete set of the state chosen for evaluation. This fact is apparent by defining the density operator ρ as follows:

$$\rho = \sum_R P_R |R\rangle\langle R|. \tag{2.3}$$

The density operator contains exactly the same information as the probability distribution P_R. The average $\langle O\rangle$ can be written

$$\langle O\rangle = \mathrm{Tr}(\rho O) \tag{2.4}$$

where the trace of an operator (hence abbreviated to Tr) is the sum of its diagonal matrix elements for any complete set of states. In particular, the expectation value for the identity operator is equal to unity by the definition of the probability $\mathrm{Tr}(\rho) = \sum_R P_R = 1$.

We can regard a pure state as a special case of a statistical mixture in which one of the probabilities P_R is equal to unity and all the remaining P_R are zero. The pure-state density operator is defined as

$$\rho_R = |R\rangle\langle R|. \tag{2.5}$$

In this case, the state is retracted on a particular state and statistical description becomes somewhat redundant. However, the concept of the density operator remains valid. Also the pure-state density operator satisfies the property of the projection operator

$$\rho_R^2 = \rho_R \tag{2.6}$$

which is easily proved from definition (2.5).

From now on, for later application, we will particularly restrict our attention to the bosonic system $(a, a^\dagger, N)$, where

$$[a, a^\dagger] = 1 \qquad [N, a^\dagger] = a^\dagger. \tag{2.7}$$

The Hilbert state $|n\rangle$ is characterized by the number operator

$$N|n\rangle = n|n\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle. \tag{2.8}$$

The ground state $|0\rangle$ is defined by the annihilation operator $a|0\rangle = 0$ and all other states are constructed from it: $|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle$.

The generators of the boson algebra can be decomposed into the magnitude N, i.e. the number operator and the phase ϕ, if we quantize in the polar coordinate (N, ϕ) of the phase space,

$$a = e_a\sqrt{N} \qquad a^\dagger = \sqrt{N}e_a^\dagger. \tag{2.9}$$

Here, the newly introduced operators $(e_a, e_a^\dagger)$ are called the exponential phase operators. To denote its relation with the a-boson, we introduce subscript a. There are various different definitions of the exponential phase operators, but we will choose the Susskind–Glogower notation [21, 22],

$$e_a e_a^\dagger = 1 \qquad e_a^\dagger e_a = 1 - |0\rangle\langle 0| \tag{2.10}$$

which is not unitary. The algebra is called the Cuntz algebra and will play an important role in the large N-expansion of the matrix model [23–25] and when calculating the anomaly [26]. Although the Susskind–Glogower notation is mathematically easy and a useful treatment, it is uncertain that it is a real physical operator which describes the nature [27]. Nevertheless we will choose the notation for its convenience.

As we see in products of the exponential phase operators, they are similar to the real phase (the unitary operator) except the vacuum. The commutation relations among $(N, e_a^\dagger, e_a)$ are derived from the boson algebra (2.7) and definition (2.9),

$$[N, e_a] = -e_a \qquad [N, e_a^\dagger] = e_a^\dagger. \tag{2.11}$$

The exponential phase operator and its ajoint shift up and down one step in the number state,

$$e_a^\dagger|n\rangle = |n+1\rangle \qquad e_a|n\rangle = |n-1\rangle. \tag{2.12}$$

We now turn to the properties of the exponential phase operators in order to find their physical meanings [20]. They are easily seen in the coherent state of the boson algebra defined by the eigenstate of the annihilation operator,

$$a|z\rangle = z|z\rangle. \tag{2.13}$$

This $|z\rangle$ is normalized, $\langle z|z\rangle = 1$, and thus expressed by the superposition of the number states:

$$|z\rangle = \mathrm{e}^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{2.14}$$

The coherent density operator ρ_c can be written as

$$\rho_c = |z\rangle\langle z|. \tag{2.15}$$

The density operator shows a Poisson distribution with respect to the number state:

$$|\langle n|z\rangle|^2 = \exp(-|z|^2)\frac{|z|^{2n}}{n!}. \tag{2.16}$$

The mean value of the number operator (or mean number in short) under this probability is $|z|^2$, i.e. the radius or the magnitude of the complex variable z,

$$\langle z|N|z\rangle = |z|^2. \tag{2.17}$$

Similarly, the expectation value of the exponential phase operator becomes the physical (real) phase in the limit of the large mean number (i.e. the classical limit),

$$\langle z|\, e_a|z\rangle = z\,\exp(-|z|^2)\sum_n \frac{|z|^{2n}}{((n+1)!n!)^{1/2}}. \tag{2.18}$$

It is not possible to evaluate the summation analytically, the asymptotic expansion is obtained for large $|z|^2$, i.e. the classical limit, with

$$\sum_n \frac{|z|^{2n}}{n!(n+1)^{1/2}} = \frac{\exp|z|^2}{|z|}\left(1 - \frac{1}{8|z|^2} + \cdots\right) \qquad |z|^2 \gg 1. \tag{2.19}$$

Thus, the expectation value of the exponential phase operator becomes

$$\langle z|e_a|z\rangle = \frac{z}{|z|}\left(1 - \frac{1}{8|z|^2} + \cdots\right) \to \mathrm{e}^{\mathrm{i}\phi}. \tag{2.20}$$

Since z is a complex number, we can decompose it into its magnitude and phase, $z = |z|\mathrm{e}^{\mathrm{i}\phi}$, then the exponential phase operator becomes the physical phase asymptotically.

We define other well known density operators for later convenience. The pure-state density operator for a number state is given by

$$\rho_n = |n\rangle\langle n|. \tag{2.21}$$

For the thermally excited state, we define the exponential distribution (a Planck distribution) for a given temperature T and a quantal energy ϵ_0 as

$$\begin{aligned}\rho_e &= (1-q^2)\sum_n q^{2n}|n\rangle\langle n| \\ &= (1-q^2)q^{2N}\end{aligned} \tag{2.22}$$

where a q-parameter is given by

$$q^2 = \exp(-\epsilon_0/k_B T). \tag{2.23}$$

This parameter will act like the q-parameter in a q-deformation. Thus we can see a physical deformation depending on the temperature T and the quantal energy ϵ_0.

The following mean values under the exponential distribution (2.22) are obtained as

$$\langle a^\dagger a\rangle = \frac{q^2}{1-q^2} \tag{2.24}$$

$$\langle a a^\dagger\rangle = \frac{1}{1-q^2}. \tag{2.25}$$

Also those related with the exponential phase operators (2.10) are obtained as follows:

$$\langle e_a^{\alpha} e_a^{\dagger\alpha} \rangle = 1 \tag{2.26}$$

$$\langle e_a^{\dagger\alpha} e_a^{\alpha} \rangle = q^{2\alpha} \tag{2.27}$$

$$\langle \theta(N-\alpha) \rangle = q^{2\alpha}. \tag{2.28}$$

Here, $\theta(x)$ is a step function and $\alpha \geqslant 0$ is taken as an integer in order to keep the structure of the number state.

3. Construction of q-bosons and the hidden transformations

We consider a system in which the algebra is described by the generators $(D_\pm, D_0)$ satisfying the formal commutation relation

$$[D_-, D_+] = D_0. \tag{3.1}$$

We will not specify the commutators between $D_\pm$ and D_0. Although they are important to close an algebra, their effect is null in the expectation value. Furthermore, we assume that two sets of independent bosons $\{(a, a^\dagger, N_a), (b, b^\dagger, N_b)\}$ contribute to the generators $D_\pm$ in the product forms. The Schwinger representation of $su(1,1)$ (or $su(2)$) is the most typical example in which a product of two independent bosons forms the new algebra. The simple product of two independent bosons has been well studied [28], so we should extend it to that of independent bosonic contributions such as boson generators themself, their exponential phase operators and combinations of the exponential phase operators and the number operators. A part of the algebra has already been treated in [19].

We choose the forms of two operators $D_\pm$ as follows:

$$D_\pm = A_\pm B_\pm. \tag{3.2}$$

$A_\pm$ and $B_\pm$ are assumed to be independent, i.e. commuting, and have the property of only a- and b-bosons, respectively. In order not to complicate matters further, we restrict $A_\pm$ and $B_\pm$ to have only one quantal number of their own:

$$\begin{aligned} &[N_a, A_\pm] = \pm A_\pm \qquad [N_b, A_\pm] = 0 \\ &[N_a, B_\pm] = 0 \qquad [N_b, B_\pm] = \pm B_\pm \\ &[A_\pm, B_\pm] = 0 \qquad [A_\mp, B_\pm] = 0. \end{aligned} \tag{3.3}$$

Here N_i, $i = a, b$ is the number operator of each boson.

From the product form (3.2) of $D_\pm$, we rewrite the commutator (3.1) as follows:

$$A_- A_+ B_- B_+ - A_+ A_- B_+ B_- = D_0. \tag{3.4}$$

We assume that its full Hilbert space of $A_\pm$ in $D_\pm$ is known. The property of the $B_\pm$ should be determined to keep the algebra of $D_\pm$ depending on the forms of D_0. Under this assumption, equation (3.4) itself is the relation for $B_\pm$. To solve it, we should take an expectation value for a density operator,

$$\langle A_- A_+ \rangle \langle B_- B_+ \rangle - \langle A_+ A_- \rangle \langle B_+ B_- \rangle = \langle D_0 \rangle. \tag{3.5}$$

We will discuss the results with respect to density operators. Two types of density operators (2.21) and (2.22) will be used in this paper to see the algebraic structure (the pure-state density operator) and the q-deformation (the exponential density operator). The coherent density operator has already been used to interpret the physical meanings of the exponential phase operator. Then relation (3.5) gives an algebraic relation depending on the forms of the density operators. Generally we use the pure-state density operator only.

We briefly discuss the algebraic solution for $B_\pm$ from (3.5). We take the pure-state density operators for both $A_\pm$ and $B_\pm$. The relation reduces to an algebraic difference equation in the variables n_a and n_b, formally written as

$$\mathcal{A}(n_a+1)\mathcal{B}(n_b+1) - \mathcal{A}(n_a)\mathcal{B}(n_b) = \mathcal{D}(n_a, n_b). \tag{3.6}$$

Since the full structure of $A_\pm$ is assumed known, we explicitly know about the form of $\mathcal{A}$. Relation (3.6) becomes a difference equation for $\mathcal{B}$. So we may find an algebraic solution for $\mathcal{B}$, and sequentially obtain the form of $B_\pm$ with respect to n_b. The $\mathcal{B}$ formally represents the square of magnitude of $B_\pm$. Since we assume from the outset that $B_\pm$ changes a quantal number of N_b, they are thus easily realized by the product of the square root of $\mathcal{B}$ and the exponential phase operator. The so-obtained solution of $B_\pm$ is a formal one which we should change into a normal form expressed by the boson generators $(b, b^\dagger)$. This can be done by using the definition of the exponential phase operator (2.9) and (2.10).

However, the situation is similar if we take an exponential density operator for $A_\pm$ and a pure-state density operator for $B_\pm$, respectively. We should choose the pure state for $B_\pm$, because their full Hilbert space is not assumed to be known. Relation (3.5) under the resulting expectation value of $A_\pm$ gives a new relation for $B_\pm$ depending on the parameter arising from the density operator. In other words, the relation for $B_\pm$ looks like the deformed algebra with respect to the algebraic solution. In reality, the forms of $B_\pm$ as the solution should not depend on the forms of the density operators, since the algebra of $(D_\pm, D_0)$ is satisfied without the density operator. The resulting algebra should not have the deformation parameter [19, 20], but, in many cases, the relation takes the form of the well known q-deformed algebra under redefinition of the parameter. As a result, we can consider the process of taking the expectation value as a tool to obtain a q-deformation physically. Also this method explicitly and directly explains why the mutual transformation in the equivalent class of the q-deformations is possible and why they take such forms. The explanation is that they are related to each other by the various transformations from the initial undeformed operators.

We now proceed to q-bosons. The q-bosons have been defined in different forms:

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = \begin{cases} 1 & \text{type I} \quad (3.7) \\ 1-q^2 & \text{type I}' \quad (3.8) \\ q^{-2N_b} & \text{type II} \quad (3.9) \\ (1-q^2)q^{-2N_b} & \text{type II}'. \quad (3.10) \end{cases}$$

Here we impose the various structural types to the q-bosons for our own convenience. These q-bosons have been introduced by different authors [1–10]. Their mutual transformations can be found in [11–13]. We now reconstruct the above types of q-bosons from the relation (3.5).

First, we proceed to the q-boson of type I by taking $A_\pm$ as the exponential phase operators and D_0 as the identity operator up to the a-vacuum,

$$A_+ = e_a^\dagger \qquad A_- = e_a \qquad D_0 = 1. \tag{3.11}$$

Relation (3.4) is rewritten as $e_a e_a^\dagger B_- B_+ - e_a^\dagger e_a B_+ B_- = 1$. The algebraic solution of $B_\pm$ is approximately the bosonic generators, i.e. $B_+ = b^\dagger$, $B_- = b$. We note that, to get an algebraic bosonic solution, D_0 should be in the form $D_0 = 1 + N_b|0_a\rangle\langle 0_a|$. Thus we say that D_0 is the identity operator up to the a-vacuum. Taking $D_0 = 1$ is *a kind of a constraint*. For a more detailed treatment of the constraint see [19].

Then what is the meaning of $D_+ = e_a^\dagger B_+ (D_- = e_a B_-)$? To see this, we recall that the exponential phase operator becomes the physical phase operator in the large mean number limit (the classical limit) of the coherent state, i.e. $\langle z|e_a|z\rangle = z/|z| + \cdots \sim e^{i\phi}$. Since $B_\pm$ is equal to the boson generators, the related generators $D_- = e_a B_-$ ($D_+ = e_a^\dagger B_+$) are similar to a global $U(1)$ gauge transformation of the normal boson generators in the classical limit of the coherent state of the a-boson. Thus we can say that the operators $D_\pm$ are the operator global $U(1)$ gauge transformation of the normal boson. $B_\pm$ is obtained from $D_\pm$ by the operator $U(1)$ gauge fixing, so $B_\pm$ is interpreted as the gauge fixed operator of $D_\pm$. Taking the expectation value can be understood as gauge fixing.

Relation (3.5) under condition (3.11), i.e. taking the expectation values for the exponential density operator of $A_\pm$ in (2.26) and (2.27), gives

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = 1. \tag{3.12}$$

Here, we put the tilde on the operator in order to represent the connection between the deformed and the original operators. As a result, $\tilde{B}_\pm$ is the q-boson of type I. We note that the q-parameter arises from the exponential distribution of the statistically mixed state. We use the density operator

$$\rho_{en_b} = (1-q^2)\sum_{n_a=0}^{\infty} q^{2n_a}|n_a, n_b\rangle\langle n_b, n_a|. \tag{3.13}$$

As a result, we can say that *the operator $U(1)$ gauge transformed boson is deformed into the q-boson of type I*.

As a second example, we choose $A_\pm$ as the boson and D_0 as the identity operator up to the b-vacuum in (3.5):

$$A_+ = a^\dagger \qquad A_- = a \qquad D_0 = 1. \tag{3.14}$$

This example is the reverse choice of the first example (3.11). The relation (3.4) is rewritten as $aa^\dagger B_-B_+ - a^\dagger a B_+B_- = 1$. Note that $D_0 = 1 + N_a|0_b\rangle\langle 0_b|$. Thus the state sum for $A_+ = a^\dagger$ ($A_- = a$) is just like that of the virtual particle state and as a result finds the gauge information only.

The algebraic solution for $B_\pm$ is the exponential phase operator,

$$B_- = e_b \qquad B_+ = e_b^\dagger. \tag{3.15}$$

This solution is easily seen from the first example (3.11).

We take the expectation value with the density operator (3.13), then consulting (2.24) and (2.25) gives the relation

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = 1 - q^2. \tag{3.16}$$

This relation shows the q-boson of type I′. As a result, *the gauge transformation, i.e. the exponential phase operator, is deformed into the q-boson of type I′*.

We now construct the q-boson of type II′, and back to type II. We take $A_\pm$ as the boson and D_0 as the identity except up to the αth state of the a-boson and the b-vacuum, i.e.

$$A_+ = a^\dagger \qquad A_- = a \qquad D_0 = \theta(N_a - \alpha). \tag{3.17}$$

Relation (3.4) is rewritten as $aa^\dagger B_-B_+ - a^\dagger a B_+B_- = \theta(N_a - \alpha)$. In order to keep the algebraic structure, we require that α is independent of n_a,

$$[N_a, \alpha] = 0. \tag{3.18}$$

The corresponding algebra looks very strange in terms of D_0, since it is equal to the identity depending on states of the algebra. This special property will be treated further after constructing the q-boson of type II′.

The algebraic solution for $B_\pm$ of (3.4) is the exponential phase operator satisfying (2.10). After taking the expectation value for the density operator (3.13) and consulting (2.24), (2.25) and (2.28), relation (3.4) gives the q-boson,

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = (1-q^2)q^{-2\alpha}. \tag{3.19}$$

This relation shows the q-boson of type II′ if we take

$$\alpha = N_b \tag{3.20}$$

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = (1-q^2)q^{-2N_b}. \tag{3.21}$$

This choice is acceptable, since $[N_a, N_b] = 0$.

From now on, we consider the meaning of this choice (3.17). Under the choice, relation (3.4) is written as

$$aa^\dagger B_-B_+ - a^\dagger a B_+B_- = \theta(N_a - \alpha). \tag{3.22}$$

We first change the nontrivial term $\theta(N_a - \alpha)$ into the identity by a proper transformation. It is just an operator similarity (i.e. adjoint) transformation of x by the αth power of the exponential phase operator e_a^α such as $e_a^{\dagger\alpha} x e_a^\alpha$. We take the operator similar transformation in which $D_0 = \theta(N_a - \alpha)$ changes into the identity operator,

$$e_a^{\dagger\alpha}\theta(N_a - \alpha)e_a^\alpha = \theta(N_a) = 1. \tag{3.23}$$

We make an operator similarity transformation on the boson algebra $(a, a^\dagger, N_a)$ and write them as $(a(\alpha), a^\dagger(\alpha), N_a(\alpha))$ with

$$a(\alpha) = e_a^{\dagger\alpha} a e_a^\alpha \qquad a^\dagger(\alpha) = e^{\dagger\alpha} a^\dagger e_a^\alpha. \tag{3.24}$$

These operators $(a(\alpha), a^\dagger(\alpha))$ form a *new* boson, so we call it the *α-adjoint boson* from now on. The relation (3.22) changes into (3.14) under the α-adjoint boson, i.e.

$$a(\alpha)a^\dagger(\alpha)B_-B_+ - a^\dagger(\alpha)a(\alpha)B_+B_- = 1. \tag{3.25}$$

The q-boson (3.21) is an operator α-adjoint transformed version of (3.16).

Let us consider more properties of the α-adjoint boson. The generators of the α-adjoint boson satisfy the boson algebra with a different vacuum,

$$[a(\alpha), a^\dagger(\alpha)] = \theta(N_a - \alpha). \tag{3.26}$$

The vacuum of the $a(\alpha)$-boson is defined in the same way as the a-boson with

$$a(\alpha)|0_a(\alpha)\rangle = 0. \tag{3.27}$$

This vacuum contains up to the αth number state of the normal boson,

$$a(\alpha)|n_a\rangle = 0 \qquad \text{if } n_a \leqslant \alpha. \tag{3.28}$$

Thus we can think of the α-vacuum as the sum of the annihilated states,

$$|0_a(\alpha)\rangle = \sum_{n_a=0}^{\alpha} c_{n_a}|n_a\rangle \tag{3.29}$$

where the c_{n_a}'s are constants that should satisfy the normalization

$$\sum_{n_a=0}^{\alpha} |c_{n_a}|^2 = 1 \tag{3.30}$$

since the vacuum is normalized to unity ($\langle 0_a(\alpha)|0_a(\alpha)\rangle = 1$). The vacuum of the $a(\alpha)$-boson forms the fundamental representation of the global $SU(\alpha+1)$ symmetry.

We define the number operator $N_a(\alpha)$ of the $a(\alpha)$-boson by

$$N_a(\alpha) = a^\dagger(\alpha)a(\alpha). \tag{3.31}$$

The number state $|n_a(\alpha)\rangle$ of the $a(\alpha)$-boson is given by

$$N_a(\alpha)|n_a(\alpha)\rangle = n_a|n_a(\alpha)\rangle \tag{3.32}$$

$$|n_a(\alpha)\rangle = \frac{1}{\sqrt{n_a(\alpha)!}}(a^\dagger(\alpha))^{n_a}|0_a(\alpha)\rangle = |n_a + \alpha\rangle \qquad n_a \geqslant 0. \tag{3.33}$$

We take the expectation value of the $a(\alpha)$-boson for the exponential density operator (2.22).

Motivated from the above operator adjoint transformation of the boson generators, we make the transformation on the exponential phase operator,

$$e_a(\alpha) = e_a^{\dagger\alpha} e_a e_a^\alpha \qquad e_a^\dagger(\alpha) = e_a^{\dagger\alpha} e_a^\dagger e_a^\alpha. \tag{3.34}$$

These exponential phase operators act like the exponential phase operator (2.10) for the vacuum $|0_a(\alpha)\rangle$,

$$e_a(\alpha)e_a^\dagger(\alpha) - e_a^\dagger(\alpha)e_a(\alpha) = |0_a(\alpha)\rangle\langle 0_a(\alpha)|. \tag{3.35}$$

They also change the quantum number by one step and satisfy the same form of commutation relations as in (2.11).

Finally we can construct the q-boson of type II by using the α-adjoint exponential phase operator. We take $A_\pm$ as the α-adjoint exponential phase operators and D_0 as the identity up to the vacuum $|0_a(\alpha)\rangle$ in (3.4),

$$A_+ = e_a^\dagger(\alpha) \qquad A_- = e_a(\alpha) \qquad D_0 = 1. \tag{3.36}$$

Then the algebraic solution for $B_\pm$ is the bosonic generators, i.e. $B_+ = b^\dagger$, $B_- = b$. Thus $D_\pm$ can be considered as the operator global $U(1)$ gauge transformation of the boson generators. Relation (3.5), under the density operator (3.13), gives

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = q^{-2\alpha}. \tag{3.37}$$

This algebra is the q-boson of type II if we choose $\alpha = N_b$, with

$$\tilde{B}_-\tilde{B}_+ - q^2\tilde{B}_+\tilde{B}_- = q^{-2N_b}. \tag{3.38}$$

As a result, the q-boson of type II is related to the gauge transformation by the adjoint transformation of the exponential phase operator.

4. $SU_q(N)$-covariant bosons and $SU(N)$ symmetry

We now proceed to extend the system (3.1) to an N-component system $(D_{+i}, D_{-i}, D_{0ij})$ which satisfies the commutation relations

$$[D_{-i}, D_{+j}] = D_{0ij} \tag{4.1}$$

in the formal sense. All other commutators are related to the ladder structure of the algebra and we will fix them in the middle of the construction of q-bosons.

Similar to the previous one-component case, we introduce two sets of N-component independent bosons $\{(a_i, a_i^\dagger, N_{ai})\ (b_j, b_j^\dagger, N_{bj})\}$ such that

$$[a_i, a_j^\dagger] = \delta_{ij} = [b_i, b_j^\dagger] \qquad i, j = 1, 2, \ldots, N$$
$$N_{ai} = a_i^\dagger a_i \qquad N_{bi} = b_i^\dagger b_i \qquad i = 1, \ldots, N. \tag{4.2}$$

The Hilbert space, characterized by the number operators (N_{ai}, N_{bi}), is spanned by pure states $|n_{ai}, \ldots, n_{bN}\rangle$.

We assume that $D_{\pm i}$ are decomposed into the a_i- and b_i-contributions,

$$D_{-i} = A_{-i}B_{-i} \qquad D_{+i} = A_{+i}B_{+i}. \tag{4.3}$$

The mutually independent $A_{\pm i}$ and $B_{\pm i}$ are fixed to the one-step operator,

$$\begin{aligned} &[N_{ai}, A_{\pm j}] = \pm\delta_{ij}A_{\pm i} \qquad [N_{bi}, B_{\pm j}] = \pm\delta_{ij}B_{\pm i} \\ &[A_{\pm i}, A_{\pm j}] = 0 \qquad [A_{\pm i}, B_{\pm j}] = 0 = [A_{\pm i}, B_{\mp j}] \qquad i \neq j. \end{aligned} \tag{4.4}$$

For simplicity, the choice of $A_{\pm i}$ is restricted to a normal boson and an exponential phase operator. Also, D_{0ij} is fixed to the identity operator and the step function to find the known q-boson. After substituting the product form of $D_{\pm i}$ into the commutator (4.1), we rewrite the relation as

$$A_{-i}A_{+j}B_{-i}B_{+j} - A_{+j}A_{-i}B_{+j}B_{-i} = D_{0ij}. \tag{4.5}$$

Here D_{0ij} is a function of the number operators (N_{ai}, N_{bi}). We take an expectation value of relation (4.5) for an exponential density operator for the a_i-boson and a pure-state density operator for the $B_{\pm i}$ state, with

$$\rho = \sum_{n_{ai}=0}^{\infty} \left(\prod_{i=1}^{N} (1 - q_i^2) q_i^{2n_{ai}} \right) |n_{ai}, n_{bi}\rangle\langle n_{ai}, n_{bi}|. \tag{4.6}$$

Then the result will be a q-deformation of the algebra of $B_{\pm i}$ as we have done in the previous section. The q-parameters are given by

$$q_i^2 = \mathrm{e}^{-\epsilon_i/k_{\mathrm{B}}T} \qquad i = 1, 2, \ldots, N. \tag{4.7}$$

Note that these q-parameters differ with the quantal energies.

We first take the exponential phase operator for $A_{\pm i}$ and the identity operator for D_{0ij} in (4.5) as

$$A_{-i} = e_{ai}(\alpha_i) \qquad A_{+i} = e_{ai}^{\dagger}(\alpha_i) \qquad D_{0ij} = \delta_{ij}. \tag{4.8}$$

Here, $e_{ai}(\alpha_i)$ is the similarity (adjoint) transformation of the exponential phase operator,

$$e_{ai}(\alpha_i) = e_{ai}^{\dagger\alpha_i} e_{ai} e_{ai}^{\alpha_i}. \tag{4.9}$$

In order for $A_{\pm i}$ to be independent of each other and $B_{\pm j}$, α_i should commute with the number operators N_{ai} and N_{bj}. Furthermore, we take them as numbers. Relation (4.5) is rewritten as $e_{ai}(\alpha_i)e_{aj}^{\dagger}(\alpha_j)B_{-i}B_{+j} - e_{aj}^{\dagger}(\alpha_j)e_{ai}(\alpha_i)B_{+j}B_{-i} = \delta_{ij}$.

The algebraic solutions for $B_{\pm i}$ in (4.5) are bosons,

$$B_{-i} = b_i \qquad B_{+i} = b_i^{\dagger}. \tag{4.10}$$

After taking the expectation value (4.5) with respect to (4.6), the nontrivial relations are seen in the same species, since the density operator (4.6) is diagonal, with

$$\tilde{B}_{-i}\tilde{B}_{+i} - q_i^2\tilde{B}_{+i}\tilde{B}_{-i} = q_i^{\alpha_i}. \tag{4.11}$$

These are relations of type II for each species. We then encounter a problem to determine the other relations between the different species consistently. To treat this problem properly, we should take a q-differential calculus and a Yang–Baxter equation into consideration [29, 30], but we leave further details to a future paper, treat only the independent cases and restrict our attention here to finding the hidden symmetries of the system.

We assume that all the α_i are equal to zero,

$$\alpha_i = 0. \tag{4.12}$$

The different species of $B_{\pm i}$ are mutually independent and commuting in relations (4.11) Thus we construct the N-component independent q-bosons of different q-deformation parameters:

$$\tilde{B}_{-i}\tilde{B}_{+j} - q_j^{2\delta_{ij}}\tilde{B}_{+j}\tilde{B}_{-i} = \delta_{ij} \qquad [N_{bi}, \tilde{B}_{\pm j}] = \pm\tilde{B}_{\pm j}\delta_{ij}. \tag{4.13}$$

These independent q-bosons are simple extensions of the q-boson of type II into N components.

We construct the $su_q(N)$ algebra by using the Schwinger method and find the hidden symmetry taking place in the process of constructing the algebra. The $su_q(N)$ algebra is given by

$$\begin{aligned} &[H_i, H_j] = 0 \qquad [E_i, F_j] = \delta_{ij}[H_i] \\ &[H_i, E_j] = A_{ij}E_j \qquad [H_i, F_j] = -A_{ij}F_j \end{aligned} \tag{4.14}$$

where $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$ and $A_{ij} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ij-1}$ is the element of the $su(N)$ Cartan matrix.

This algebra has only one deformation parameter. In order to construct $su_q(N)$ by combining the two independent q-bosons of (4.13) in the product form, we should require all q_j-values to be equal, with

$$q_1^2 = q_2^2 = \cdots = q_N^2 \equiv q^2. \tag{4.15}$$

The same q-parameter means that the quantal energies of the Hamiltonian in the density operator (4.6) should be equal, that is $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_N = \epsilon$. Thus the Hamiltonian becomes

$$H = \epsilon \sum a_i^\dagger a_i. \tag{4.16}$$

The Hamiltonian in the density operator (4.6) has a global $SU(N)$ symmetry,

$$\begin{aligned} &a_i' = \sum_j U_{ij}a_j \qquad a_i'^\dagger = \sum_j U_{ij}^\dagger a_j'^\dagger \\ &\sum_k U_{ik}U_{kj}^\dagger = \sum_k U_{ik}^\dagger U_{kj} = \delta_{ij}. \end{aligned} \tag{4.17}$$

Under this symmetry, the q-boson (4.13) is changed into a q-parameter algebra:

$$\tilde{B}_{-i}\tilde{B}_{+j} - q^{2\delta_{ij}}\tilde{B}_{+j}\tilde{B}_{-i} = \delta_{ij} \qquad [N_{bi}, \tilde{B}_{\pm j}] = \pm\tilde{B}_{\pm j}\delta_{ij}. \tag{4.18}$$

This algebra has the global $SU(N)$ transformation (4.17) as its hidden symmetry. Also the Chevalley basis of $su_q(N)$ are given, in [4], by

$$\begin{aligned} &H_i = N_i - N_{i+1} \qquad E_i = \tilde{B}_{+i}\tilde{B}_{-(i+1)} \qquad F_i = \tilde{B}_{+(i+1)}\tilde{B}_{-i} \\ &i = 1, \ldots, N-1. \end{aligned} \tag{4.19}$$

As a result, we find the relation between the global $SU(N)$ symmetry and the Schwinger realization of $su_q(N)$.

As a second example, we consider the $SU_q(N)$-covariant bosons [5], where

$$\begin{aligned} &\tilde{B}_{-i}\tilde{B}_{+i} - q^2\tilde{B}_{+i}\tilde{B}_{-i} = q^{2\sum_{j=1}^{i-1} N_j} \\ &\tilde{B}_{-i}\tilde{B}_{-j} = q\tilde{B}_{-j}\tilde{B}_{-i} \qquad i < j \\ &\tilde{B}_{-i}\tilde{B}_{+j} = q\tilde{B}_{+j}\tilde{B}_{-i} \qquad i \neq j. \end{aligned} \tag{4.20}$$

The algebra can be rewritten by the $SU(N)$ R-matrix,

$$R = q\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} \tag{4.21}$$

where e_{ij} is the $N \times N$ matrix with entry one at position (i, j) and zero elsewhere. Using the notation $R = R_{ij,kl} e_{ik} \otimes e_{jl}$, after a minor modification, relations (4.20) are rewritten as

$$\begin{aligned}\tilde{B}_{-i}\tilde{B}_{-j} &= q^{-1} R_{ij,kl} \tilde{B}_{-l}\tilde{B}_{-k} \\ \tilde{B}_{+i}\tilde{B}_{+j} &= q^{-1} R_{lk,ij} \tilde{B}_{+k}\tilde{B}_{+l} \\ \tilde{B}_{-i}\tilde{B}_{+j} &= \delta_{ij} + q R_{ki,jl} \tilde{B}_{+k}\tilde{B}_{-l}.\end{aligned} \tag{4.22}$$

We now derive the algebra (4.20). We choose new forms of $A_\pm$ and D_{0ij} in (4.5) as follows:

$$A_{-i} = e_{ai} \qquad A_{+i} = e_{ai}^\dagger \qquad D_{0ij} = \theta\left(N_{ai} - \sum_{k=1}^{i-1} N_{bk}\right)\delta_{ij}. \tag{4.23}$$

Relation (4.5) can be written as

$$e_{ai}e_{aj}^\dagger B_{-i}B_{+j} - e_{aj}^\dagger e_{ai} B_{+j}B_{-i} = \delta_{ij}\theta\left(N_{ai} - \sum_{k=1}^{i-1} N_{bk}\right). \tag{4.24}$$

We take the expectation value for the density operator (4.6) and obtain the non-trivial relation only for $i = j$. We also require that the a_i-bosons are all equivalent, i.e. their Hilbert space has the global $SU(N)$ symmetry as in the independent q-bosons. The non-trivial parts under the expectation values of (4.24) are given by

$$\tilde{B}_{-i}\tilde{B}_{+i} - q^2 \tilde{B}_{+i}\tilde{B}_{-i} = q^{2\sum_{j=1}^{i-1} N_j}. \tag{4.25}$$

The different q-bosons may be mutually commuting if we ignore the tower-like dependence on the number operators (4.25), but the right-hand side of (4.25) prevents them from being independent q-bosons. We introduce independent (commuting) operators which take the right-hand side of (4.25) into a constant, with

$$\hat{B}_{\pm i} = q^{\sum_{k<i} N_k} \tilde{B}_{\pm i}. \tag{4.26}$$

Then we obtain the algebra (4.18). We can thus treat them as independent in order to fix the relation between different species:

$$\hat{B}_{\pm i}\hat{B}_{\pm j} = \hat{B}_{\pm j}\hat{B}_{\pm i} \qquad \hat{B}_{\pm i}\hat{B}_{\mp j} - q^{2\delta_{ij}}\hat{B}_{\mp j}\hat{B}_{\pm i} = \delta_{ij}. \tag{4.27}$$

These relations can be rewritten in terms of the original operators $\tilde{B}_{\pm i}$. We then obtain the q-commuting property

$$\begin{aligned}&\tilde{B}_{-i}\tilde{B}_{-j} = q\tilde{B}_{-j}\tilde{B}_{-i} \qquad \tilde{B}_{+i}\tilde{B}_{+j} = q\tilde{B}_{+j}\tilde{B}_{+i} \qquad i < j \\ &\tilde{B}_{-i}\tilde{B}_{+j} = q\tilde{B}_{+j}\tilde{B}_{-i} \qquad i \neq j.\end{aligned} \tag{4.28}$$

After combining (4.25) and (4.28), we obtain the fundamental representation of $SU_q(N)$. In reality, we needed to build in the commutatitivity of the transformed operators of the different species by hand. A more systematic and general approach requires the q-differential calculus [29, 30].

We now consider the meaning of the choice of (4.24) as done in the previous section. Introduce the operator similar transformation

$$e_{ai}(\alpha_i) = e_{ai}^{\dagger\alpha_i} e_{ai} e_{ai}^{\alpha_i} \qquad e_{ai}^\dagger(\alpha_i) = e_{ai}^{\dagger\alpha_i} e_{ai}^\dagger e_{ai}^{\alpha_i} \tag{4.29}$$

where

$$\alpha_i = \sum_{k=1}^{i-1} N_{bk}. \tag{4.30}$$

Then the system (4.24) is changed into

$$e_{ai}(\alpha_i)e^{\dagger}_{aj}(\alpha_j)B_{-i}B^{\dagger}_{+j} - e^{\dagger}_{aj}(\alpha_j)e_{ai}(\alpha_i)B_{+j}B_{-i} = \delta_{ij}. \tag{4.31}$$

This relation is equivalent to (4.8) except for the α-adjoint exponential phase operators.

As a result, the $SU_q(N)$-covariant bosons are related to the adjoint transformation of the exponential phase operator.

5. Discussions and conclusions

We see that the averaging method is a very convenient way to find the difference and similarity between the different forms of q-bosons, but, to obtain our results, we restricted the product of two commuting generators to satisfy the algebra which is different from the product algebra up to their vacuum structure. This is a kind of constraint in which the q-boson is based, so we need to consider it further on the basis of the Hopf algebra [16, 17]. Although we were able to extend our ideas to the multicomponent system and find the hidden symmetry, it still remains to fully consider it using the machinery of the q-calculus [29, 30].

There are many different q-bosons, not treated here. However, they can be included in our method by relaxing and modifying the conditions given here. As a simple example, we can construct, by using a similar method, the q-fermion [6, 9],

$$\tilde{C}\tilde{C}^{\dagger} + q^2\tilde{C}^{\dagger}\tilde{C} = 1.$$

More interesting things appear when the density operator is fermionic and this system is related to extending the real q-parameter into a complex variable.

Various transformations using the exponential phase operator show some interesting properties, since it is a $U(1)$ phase in the classical limit. We can thus obtain an operator global $U(1)$ gauge transformation (see the q-boson of type I) by taking products with an exponential phase operator. We can also define an operator adjoint (similarity) transformation, but such transformations are not unitary, since the exponential phase operator is not unitary. In particular, the adjoint transformation of a boson algebra (3.26) has the shifted vacuum of a global $SU(\alpha)$ symmetry (3.29).

In order to generate a non-abelian gauge transformation, we need a non-abelian phase operator that is still not defined. More generally, it will also be worthwhile extending the above concepts into field theory [26] and the matrix model [23–25].

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